Photometry, I have discovered, is more difficult than I originally thought. In Astro 1xx, we all learn the equation,

\[ m_1 - m_2 = -2.5 \log(F_1/F_2), \]

where \( m_1 \) and \( m_2 \) are the apparent magnitudes of two objects and \( F_1 \) and \( F_2 \) are the “fluxes” of the two objects. This equation is deceptively simple. In practice, we have to worry about things such as instrumental magnitudes, zero points, color corrections, extinction corrections, etc. As I read more about photometry, I learned that much of my confusion was caused by my utter lack of understanding of heterochromatic extinction. Therefore, this is my attempt at trying to explain photometry to myself, especially extinction and color corrections, and as such, is not a cookbook on how to perform photometry but more an exposition on the underlying theory of photometry.

§1 deals with basics definitions. §2 deals with narrow-band photometry while §3, the heart of this work, deals with broad-band photometry. Much of what I say in this work is a compilation of the work by Young and King. I have listed the references at the end and highly recommend reading them.

1 Basics

1.1 “Flux”

I specifically put “flux” in quotes in the introduction because we have to be careful with this word. If \( F \) is a flux density (quantity time\(^{-1}\) area\(^{-1}\) wavelength\(^{-1}\)), then the magnitudes in Equation 1 are said to be monochromatic magnitudes while if \( F \) is a flux (quantity time\(^{-1}\) area\(^{-1}\)) then the magnitudes given in Equation 1 are said to be heterochromatic magnitudes. Although you do find references to monochromatic magnitudes in some of the text (I will use them too), and they do exist in the literature, most published magnitudes are heterochromatic. Besides, if you have the flux density at a particular wavelength, why would you want to convert it to a magnitude?

1.2 Photon Flux Density

Let the spectral energy distribution of the star as seen by the telescope be given by \( f_\lambda(\lambda) \) which has units \( [f_\lambda] = \text{quantity time}^{-1} \text{area}^{-1} \text{wavelength}^{-1} \). Note that if we are on the ground, \( f_\lambda(\lambda) \) must allow for the effects of atmospheric absorption. Since we deal mostly with photon counting detectors, \( \text{quantity}=\text{photons} \) and \( f_\lambda \) is therefore the photon flux density. This photon flux density can be computed as,
\[ p_\lambda(\lambda) = \frac{\lambda}{hc} e_\lambda(\lambda). \]  

where \( e_\lambda(\lambda) \) is the energy flux density. We will use the notation \( f_\lambda \) throughout this work to be completely general so it is the readers responsibility to known whether their detector counts photons or measures energy.

### 1.3 Alternate Magnitude Equations

Equation 1 can also be written in two equivalent forms,

\[
m = -2.5 \log(F) + ZP \quad (3)
\]

\[
m = m^{\text{inst}} + ZP \quad (4)
\]

where ZP is the zero-point of the system defined by some standard and \( m^{\text{inst}} \) is the instrumental magnitude. So from now on, we will work in instrumental magnitudes and we just have to remember to add the zero-point if we want calibrated data.

### 1.4 Photometric Measurement

A photometric measurement consists of integrating a star’s observed spectrum, \( f_\lambda(\lambda) \), over a system response function, \( S(\lambda) \), to produce a flux, \( F \), and is given by the equation,

\[
F = \int_0^\infty f_\lambda(\lambda) S(\lambda) d\lambda. 
\]

(5)

\( S(\lambda) \) includes the filter transmission and the instrument throughput (which includes both the telescope and camera efficiencies). You can think of \( S(\lambda) \) as the probability that a photon of wavelength \( \lambda \) will actually be detected by the detector.

### 2 Narrow-Band Filters

To begin, consider an idealized filter which can be described by a delta function centered at \( \lambda_0 \):

\[
S(\lambda) = \delta(\lambda - \lambda_0) 
\]

(6)

### 2.1 Exo-Atmospheric Magnitudes

Now substituting Equation 6 into Equation 5 our instrumental magnitude becomes,

\[
m^{\text{inst}}(\lambda_0) = -2.5 \log f_\lambda(\lambda_0) 
\]

(7)

So that the actually magnitude of the object is,
\[ m(\lambda_0) = m_{\text{inst}}(\lambda_0) + ZP \]  

This may seem trivial (and it is!) but it is important to have a baseline to compare with in later sections.

### 2.2 Ground-Based Magnitudes

Now let’s include the effect of absorption due to the Earth’s atmosphere. The flux received by the telescope, \( f_\lambda(\lambda) \), is now a function of the airmass at which the object is observed and is given by,

\[ f_\lambda(\lambda, X) = F_\lambda(\lambda) T(\lambda, X) = F_\lambda(\lambda) \exp(-X\tau(\lambda)) \]  

where \( F_\lambda(\lambda) \) is the flux density of the object outside the Earth’s atmosphere, \( \tau(\lambda) \) is the optical depth at an airmass of 1 and \( X \) is the airmass of the object. Substituting into Equations 5 and then converting to magnitudes we obtain,

\[
m_{\text{inst}}(\lambda_0, X) = \quad -2.5 \log F_\lambda(\lambda_0) - 2.5 \log [\exp(-X\tau(\lambda_0))] \\
= m_{\text{inst}}(\lambda_0, 0) - 2.5 \log e \ln[\exp(-X\tau(\lambda_0))] \\
= m_{\text{inst}}(\lambda_0, 0) + 1.086X\tau(\lambda_0)
\]

If we rearrange a bit we obtain the instrumental magnitude of the object above the atmosphere:

\[
m_{\text{inst}}(\lambda_0, 0) = m_{\text{inst}}(\lambda_0, X) - 1.086X\tau(\lambda_0)
\]

The actual magnitude of the object above the atmosphere is then given by,

\[
m(\lambda_0, 0) = m_{\text{inst}}(\lambda_0, 0) + ZP \\
= m_{\text{inst}}(\lambda_0, X) - 1.086X\tau(\lambda_0) + ZP
\]

So we see that when observing from the ground with narrow band filters, we simply have to add a correction term which is proportional to the airmass at which the object was observed to obtain the actual magnitude of the object. Of course, this equation is only strictly true for narrow band filters.

### 3 Broad-Band Filters

#### 3.1 Isophotal Wavelength

Before we discuss broad-band photometric measurements, we need to describe a useful quantity called the isophotal wavelength. If we assume that \( f_\lambda(\lambda) \) is continuous and that \( S(\lambda) \) does not change sign then a generalization of the mean value theorem states that,
\[ f_\lambda(\lambda_{iso}) = \langle f_\lambda(\lambda) \rangle \]
\[ = \frac{\int f_\lambda(\lambda)S(\lambda)d\lambda}{\int S(\lambda)d\lambda} \tag{16} \]
or in other words, the flux density at the wavelength \( \lambda_{iso} \), \( f_\lambda(\lambda_{iso}) \), equals the average flux density as defined in Equations 17. Equation 17 can also be written as,
\[ F = f_\lambda(\lambda_{iso}) \int S(\lambda)d\lambda \tag{18} \]

If we convert this flux to an instrumental magnitude we obtain,
\[ m_{inst}^{\lambda_{iso}} = -2.5 \log \left[ f_\lambda(\lambda_{iso}) \int S(\lambda)d\lambda \right] \tag{19} \]
\[ = -2.5 \log f_\lambda(\lambda_{iso}) - 2.5 \log \int S(\lambda)d\lambda \tag{20} \]
\[ = m_{inst}(\lambda_{iso}) + S \tag{21} \]

where we have defined \( S \) to be,
\[ S = -2.5 \log \int S(\lambda)d\lambda \tag{22} \]

So we see that a heterochromatic magnitude \( m_{inst}^{\lambda_{iso}} \) is given by the monochromatic magnitude at the isophotal wavelength \( m_{inst}(\lambda_{iso}) \) plus \( S \).

### 3.2 Approximations

If the filter profile can not be described as a delta function, things get more complicated. In the following section, we will expand the spectral energy distribution around the centroid of the system response curve. After all is said and done, we will have to add correction terms onto Equation 13 to account for width of the filter. As one might expect, these correction terms are proportional to the width of the filter.

So lets assume the observed spectral energy distribution of the star \( f_\lambda(\lambda) \) is smooth enough that we can expand it around some wavelength \( \lambda_0 \) in a Taylor series:
\[ f_\lambda(\lambda) = f_\lambda(\lambda_0) + \frac{(\lambda - \lambda_0)}{1!} f'_\lambda(\lambda_0) + \frac{(\lambda - \lambda_0)^2}{2!} f''_\lambda(\lambda_0) + \cdots \tag{23} \]

Equation 5 then becomes,
\[ F = f_\lambda(\lambda_0) \int S(\lambda)d\lambda + f'_\lambda(\lambda_0) \int \frac{(\lambda - \lambda_0)}{1!} S(\lambda)d\lambda + f''_\lambda(\lambda_0) \int \frac{(\lambda - \lambda_0)^2}{2!} S(\lambda)d\lambda + \cdots \tag{24} \]
if we define the $\lambda_0$ and $\mu^2$ to be,

$$\lambda_0 = \frac{\int \lambda S(\lambda) d\lambda}{\int S(\lambda) d\lambda}, \quad \mu^2 = \frac{\int (\lambda - \lambda_0)^2 S(\lambda) d\lambda}{\int S(\lambda) d\lambda}$$

(25)

then Equation 5 becomes,

$$F = \left[ f(\lambda_0) + \frac{\mu^2}{2} f''(\lambda_0) \right] \int S(\lambda) d\lambda$$

(26)

$$= \left[ f(\lambda_0) \left( 1 + \frac{\mu^2}{2} \frac{f''(\lambda_0)}{f(\lambda_0)} \right) \right] \int S(\lambda) d\lambda.$$  

(27)

$\lambda_0$ is known as the *mean wavelength* and $\mu^2$ is proportional to the width of the filter.

### 3.3 Exo-Atmospheric Magnitudes

Converting Equation 27 into an instrumental magnitude we obtain,

$$m_{\text{inst}} = -2.5 \log f(\lambda_0) - 2.5 \log \left( 1 + \frac{\mu^2}{2} \frac{f''(\lambda_0)}{f(\lambda_0)} \right) - 2.5 \log \int S(\lambda) d\lambda$$

(28)

$$= m_{\text{inst}}(\lambda_0) - 1.086 \ln \left( 1 + \frac{\mu^2}{2} \frac{f''(\lambda_0)}{f(\lambda_0)} \right) + S$$

(29)

$$= m_{\text{inst}}(\lambda_0) - 1.086 \left( \frac{\mu^2}{2} \frac{f''(\lambda_0)}{f(\lambda_0)} \right) + S$$

(30)

where we have used the approximation $\ln(1 + x) \simeq x$. The second term is a correction term which shifts the monochromatic magnitude at the mean wavelength $m_{\text{inst}}(\lambda_0)$ to the monochromatic magnitude at the isophotal wavelength $m_{\text{inst}}(\lambda_{iso})$ (see Equation 21). The correction term involves both the spectral energy distribution and its second derivative and is proportional the width of the filter.

Finally, we obtain for the exo-atmospheric magnitude,

$$m = m_{\text{inst}} + ZP$$

(31)

$$= m_{\text{inst}}(\lambda_0) - 1.086 \left( \frac{\mu^2}{2} \frac{f''(\lambda_0)}{f(\lambda_0)} \right) + S + ZP$$

(32)

This equation should be compared with Equation 8.

### 3.4 Ground-Based Magnitudes

Now we are going to include the effects of atmospheric extinction in our approximation. Of course this requires that the extinction as a function of wavelength $T(\lambda, X)$ be smooth since our expansion
of \( f_\lambda(\lambda) \) in §3.2 requires \( f_\lambda(\lambda) \) be smooth (see Equation 9). As we shall see in a later section, this assumption is valid in the visible but is not in the infrared. If you want to skip a lot of algebra, go to Equation 66.

We have to evaluate the term in parentheses in Equation 30. So,

\[
\begin{align*}
  f_\lambda(\lambda) &= F_\lambda(\lambda)T(\lambda) \quad (33) \\
  f' &= F'T + T'F \quad (34) \\
  f'' &= F''T + 2T'F' + FT'' \quad (35)
\end{align*}
\]

so,

\[
\frac{f''}{f} = \frac{F''}{F} + \frac{2T'F'}{FT} + \frac{T''}{T} \quad (36)
\]

Next we need to evaluate \( T \) so,

\[
\begin{align*}
  T &= \exp(-X\tau) \quad (37) \\
  T' &= -X\tau' \exp(-X\tau) \quad (38) \\
  T'' &= -\tau''X \exp(-X\tau) + (X\tau')^2 \exp(-X\tau) \quad (39)
\end{align*}
\]

So,

\[
\frac{f''}{f} = \frac{F''}{F} - \frac{2\tau'XF'}{F} - \tau''X + (X\tau')^2
\]

\[
= \frac{F''}{F} + X \left( X\tau'^2 - \frac{2\tau'F'}{F} - \tau'' \right) \quad (41)
\]

Now the next step will seem a bit weird but we are going to try to convert the derivatives to logarithmic derivatives (see Appendix A) so,

\[
\frac{f''}{f} = \frac{F''}{F} + X \left[ X \left( \frac{\lambda}{\tau} \right)^2 \left( \frac{\tau}{\lambda} \right)^2 \tau'^2 - 2 \left( \frac{\lambda}{\tau} \right) \left( \frac{\tau}{\lambda} \right) \tau' \left( \frac{\lambda}{\tau} \right) \frac{F'}{F} - \left( \frac{\tau}{\lambda^2} \right) \left( \frac{\lambda^2}{\tau} \right) \tau'' \right] \quad (42)
\]

Now let (see Appendix A),

\[
a = \frac{d^2 \ln \tau}{d \ln \lambda^2}, \quad b = -\frac{d \ln \tau}{d \ln \lambda}, \quad c = \frac{d \ln F}{d \ln \lambda} \quad (43)
\]

and we can rewrite Equation 41 as,

\[
\frac{f''}{f} = \frac{F''}{F} + X \left[ X \left( \frac{\tau}{\lambda} \right)^2 b^2 + 2 \left( \frac{\tau}{\lambda} \right) bc - \frac{\tau}{\lambda^2} (a + b^2 + b) \right] \quad (44)
\]
Now we can insert this into Equation 30 and we obtain,

\[ m_{\text{inst}}(X) = m_{\text{inst}}(\lambda_0, 0) + 1.086\tau_0X - 1.086 \left( \frac{\mu^2 F''}{F} + \frac{\mu^2}{2}X[\cdots] \right) + S \]  

(45)

\[ = m_{\text{inst}}(\lambda_0, 0) - 1.086 \left( \frac{\mu^2 f''}{f_0} \right) + 1.086 (X[\tau_0 + \cdots]) + S \]  

(46)

where for brevity, we have denoted any function \( g(\lambda) \) that is evaluated at \( \lambda_0, g_0 \). We now investigate the term in brackets.

\[
\begin{bmatrix}
\tau_0 - X \frac{\mu^2}{2} \left( \frac{\tau_0}{\lambda_0} \right)^2 b^2 - 2\frac{\mu^2 \tau_0}{\lambda_0^2} bc + \frac{\mu^2}{2} \frac{\tau_0}{\lambda_0^2} (a + b^2 + b) \\
\end{bmatrix}
\]  

(47)

\[
= \left[ \tau_0[1 + \frac{\mu^2}{2} \frac{\tau_0}{\lambda_0^2} (a + b^2 + b)] - \frac{\mu^2}{2\lambda_0^2} b\tau_0(\tau_0bX + 2c) \right] 
\]  

(48)

\[
= \left[ k' - \frac{\mu^2}{2\lambda_0^2} b\tau_0(\tau_0bX + 2c) \right] 
\]  

(49)

where \( k' \) equals,

\[
k' = \tau_0[1 + \frac{\mu^2}{2} \frac{\tau_0}{\lambda_0^2} (a + b^2 + b)] 
\]  

(50)

The second term in the above equation corrects the extinction \( \tau(\lambda_0) \) to \( \tau(\lambda_{iso}) \). This term plays the same role as the second term in Equation 30.

Now let’s try to estimate the derivatives,

\[
c = \frac{d \ln F}{d \ln \lambda} = \frac{d \log F}{d \log \lambda} \approx \frac{\log(F_0/F_1)}{\log(\lambda_0/\lambda_1)} = \frac{0.4(m_1 - m_0)}{\log(\lambda_0/\lambda_1)} 
\]  

(51)

or

\[
c = \frac{0.4C}{\log(\lambda_0/\lambda_1)} 
\]  

(52)

where \( C \) is the color index \((m_1 - m_0)\) of the star.

\[
b = -\frac{d \ln \tau}{d \ln \lambda} = -\frac{\lambda_0}{\tau_0} \frac{d \tau}{d \lambda} \approx -\frac{\lambda_0}{\tau_0} \left( \frac{\tau_0 - \tau_1}{\lambda_0 - \lambda_1} \right) \approx \frac{\lambda_0}{\tau_0} \left( \frac{\tau_1 - \tau_0}{\lambda_0 - \lambda_1} \right) 
\]  

(53)

or

\[
b = \frac{\lambda_0}{(\lambda_0 - \lambda_1)} \left( \frac{\Delta \tau}{\tau_0} \right). 
\]  

(54)

where \( \Delta \tau \) is the color index \((\tau_1 - \tau_0)\) of the atmospheric extinction. Now we substitute into Equation 49 and obtain,
\[
[ ] = \left[ k' - \frac{\mu^2 \Delta \tau}{2 \lambda_0 (\lambda_0 - \lambda_1)} \left( \frac{\lambda_0 X \Delta \tau}{\lambda_0 - \lambda_1} + \frac{0.8C}{\log(\lambda_0/\lambda_1)} \right) \right] \quad (55)
\]
\[
= \left[ k' - \frac{0.4 \mu^2 \Delta \tau}{\lambda_0 (\lambda_0 - \lambda_1) \log(\lambda_0/\lambda_1)} \left( \frac{\log(\lambda_0/\lambda_1) \lambda_0 X \Delta \tau}{0.8(\lambda_0 - \lambda_1)} + C \right) \right] \quad (56)
\]
Now let \( W \) equal,
\[
W = \frac{0.4 \mu^2}{\lambda_0 (\lambda_0 - \lambda_1) \log(\lambda_0/\lambda_1)} \quad (57)
\]
Note that \( W \) is proportional to the width of the filter. Equation 56 becomes
\[
[ ] = \left[ k' - W \Delta \tau \left( \frac{\log(\lambda_0/\lambda_1) \lambda_0 X \Delta \tau}{0.8(\lambda_0 - \lambda_1)} + C \right) \right] \quad (58)
\]
Ok one more approximation assuming \((\lambda_0 - \lambda_1) << \lambda_0\),
\[
\log \left( \frac{\lambda_0}{\lambda_1} \right) = \log e \ln \left( \frac{\lambda_0}{\lambda_1} \right) = 0.434 \ln \left[ 1 + \frac{\lambda_0 - \lambda_1}{\lambda_1} \right] \quad (59)
\]
\[
\approx 0.434 \left[ \frac{\lambda_0 - \lambda_1}{\lambda_1} \right] \approx 0.434 \left[ \frac{\lambda_0 - \lambda_1}{\lambda_0} \right] \quad (60)
\]
substituting in Equation 58 we have,
\[
[ ] = \left[ k' - W \Delta \tau \left( \frac{X \Delta \tau}{1.84} + C \right) \right] \quad (61)
\]
This equation can be compared to Equation 3.1.57 of Young (1974). He obtains a value of 2 for the constant in the denominator while we obtain a constant of 1.84. This can be traced back to the form the extinction function \( T(\lambda, X) \) used to derive this equation.

Now we can substitute this into Equation 46 and we obtain,
\[
m_{\text{inst}}(X) = m_{\text{inst}}(\lambda_0, 0) - 1.086 \left( \frac{\mu^2 f_0''}{2 f_0} \right) + 1.086X \left[ k' - W \Delta \tau \left( \frac{X \Delta \tau}{1.84} + C \right) \right] + S \quad (62)
\]
\[
= m_{\text{inst}}(0) + 1.086X \left[ k' - W \Delta \tau \left( \frac{X \Delta \tau}{1.84} + C \right) \right] \quad (63)
\]

or
\[
m(0) = m_{\text{inst}}(0) + ZP \quad (64)
\]
\[
= m_{\text{inst}}(X) - 1.086X \left[ k' - W \Delta \tau \left( \frac{X \Delta \tau}{1.84} + C \right) \right] + ZP \quad (65)
\]
We see that besides the monochromatic extinction at the isophotal wavelength $k'$, we have a correction term which is proportional to the width of the filter, the color of the star $C$ and the color of the atmospheric extinction $\Delta \tau$. Let us investigate the extinction term a bit more.

3.5 Heterochromatic Extinction

Let us rearrange Equation 66 as,

$$m(X) = m(0) + 1.086X(k' - W\Delta \tau C) - \frac{1.086WX^2\Delta \tau^2}{1.84}$$ \hspace{1cm} (67)

which is the equation of a parabola. The deviation from a straight line is proportional to the $W$ and $\Delta \tau$. So if the extinction varies as a function of wavelength within the bandpass ($\Delta \tau \neq 0$), the extinction curve is non-linear.

Physically, the slope of the extinction curve varies because the wavelengths with the largest monochromatic extinctions are removed from the spectrum at the lowest airmasses (steep slope) leaving only the wavelengths with the lowest monochromatic extinctions to affect the curve at higher airmasses (shallow slope). This effect, which exists whether the above analysis holds or not, was discovered by Forbes (1892) and is eponymously known as the Forbes effect.

In the visible, extinction is generally a smooth function of wavelength so the assumption that $f_\lambda(\lambda)$ is smooth is a good one and we see that the heterochromatic extinction is correlated with the stellar color (see Equation 61). Therefore, in principle we can correct for extinction if we know the color of the object. In the infrared, extinction is not a smooth function of wavelength since it is dominated by the vibrational-rotational bands of molecules. Large and small extinction coefficients are mixed together within a band so that the above analysis does not hold. In addition, many of the lines which cause extinction are saturated by the time an airmass of 1 is reach so the extinction curve is heavily curved between an airmass of 0 and 1.

For example, Figure 1 shows an airmass plot which shows the Forbe's effect. Using observed data (airmass $> 1$) the observer will generally underestimate the magnitude of the object if the Forbes effect is ignored (straight line).

Therefore,

**Determining exo-atmospheric IR magnitudes is difficult**

As a result, most IR magnitudes are given at an airmass of 1 (e.g., 2MASS, UKIRT faint standards). This effect has also been seen in the $U$ band (Young 1974).

Finally, if we ignore the Forbes effect, and treat $W\Delta \tau$ as a constant $k''$, then we arrive the equation that is often used for heterochromatic extinction,

$$k = (k' - k''C)$$ \hspace{1cm} (68)
Figure 1: Airmass plot showing the Forbes effect.
 Appendix A: Logarithmic Derivatives

The first logarithmic derivative is given by,

\[
\frac{d \ln x}{d \ln y} = \frac{d \ln x}{d \ln y} \cdot \frac{1}{\frac{d \ln x}{d \ln y}} = \frac{dx}{x \ dy} = \frac{y \ dx}{x \ dy}
\]

(69)

The second logarithmic derivative is given by,

\[
\frac{d^2 \ln x}{d \ln y^2} = \frac{d}{d \ln y} \left( \frac{d \ln x}{d \ln y} \right) = \frac{d}{d \ln y} \left( \frac{y \ dx}{x \ dy} \right)
\]

(70)

which is equal to,

\[
\frac{d^2 \ln x}{d \ln y^2} = \frac{y \ dx}{x \ dy} - \left( \frac{y}{x} \right)^2 \left( \frac{dx}{dy} \right)^2 + \frac{y^2 \ d^2 x}{x \ dy^2}
\]

(71)

or finally,

\[
\frac{d^2 \ln x}{d \ln y^2} = \frac{y^2 \ d^2 x}{x \ dy^2} + \frac{d \ln x}{d \ln y} - \left( \frac{d \ln x}{d \ln y} \right)^2
\]

(72)
References

Forbes, J. D. 1892, Phil. Trans., 132, 225

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