The overall goal of this note is to understand the covariance terms in the error propagation formula. Most of what I am going to say is a rehash of the work by Kai Oliver Arras. Our goal is to determine the uncertainty (variance) of a quantity $z$ that is a function of $m$ random variables $q^{(1)}, q^{(2)}, \ldots, q^{(m)}$ whose distribution functions we know. In other words, we wish to know the distribution of $z$ values, $P(z)$, given $z = f(q^{(1)}, q^{(2)}, \ldots, q^{(m)})$. If $f$ is non-linear, then $P(z)$ can be very complex. Therefore, we wish to derive an approximation for $P(z)$.

1 1-D Case

Consider the function $f(q)$ shown in Figure 1. We would like to known how the distribution of $q$ values “propagates” through the function $f$ to produce the distribution of $z$ values $P(z)$. If the function is highly non-linear, the shaded region in the distribution of $q$ values will be mapped nonuniformly into $P(z)$ resulting in a distorted and asymmetric distribution. We will in the end, however, characterize $P(z)$ by its mean and variance no matter what its shape.

![Figure 1: 1-D error propagation.](dmtwww.epfl.ch/ist/asl/publications/arrasTR9801R3.pdf)
\[ z = f(q) \approx f(\mu_q) + \frac{\partial f}{\partial q}_{\mu_q} (q - \mu_q) \]  

This equation represents the dashed line in Figure 1.

We are now in a position to determine the mean and variance of the output \( z \) value using the standard formulas for the mean and variance of a set of data, namely,

\[
\mu = \frac{1}{n} \sum_{i=1}^{n} x_i \quad (2)
\]

\[
\sigma^2_{\mu} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2 \quad (3)
\]

The mean value of \( P(z) \) is given by,

\[
\mu_z = \frac{1}{n} \sum_{i=1}^{n} \left( f(\mu_q) + \frac{\partial f}{\partial q}_{\mu_q} (q_i - \mu_q) \right) \quad (4)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} f(\mu_q) + \frac{1}{n} \frac{\partial f}{\partial q}_{\mu_q} \sum_{i=1}^{n} (q_i - \mu) \quad (5)
\]

\[
= f(\mu_q) \quad (6)
\]

since by the definition of the mean value, \( \sum_{i=1}^{n} (q_i - \mu_q) = 0 \).

The variance of \( P(z) \) is given by,

\[
\sigma^2_z = \frac{1}{n} \sum_{i=1}^{n} \left( f(\mu_q) + \frac{\partial f}{\partial q}_{\mu_q} (q_i - \mu_q) - \mu_z \right)^2 \quad (7)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\partial f}{\partial q}_{\mu_q} (q_i - \mu_q) \right)^2 \quad (8)
\]

\[
= \left( \frac{\partial f}{\partial q}_{\mu_q} \right)^2 \frac{1}{n} \sum_{i=1}^{n} (q_i - \mu_q)^2 \quad (9)
\]

\[
= \left( \frac{\partial f}{\partial q}_{\mu_q} \right)^2 \sigma^2_q \quad (10)
\]

So we have the following two equations to propagate the error of a value \( q \) through a function \( z = f(q) \),
\[ \mu_z = f(\mu_q), \quad \sigma_z^2 = \left( \frac{\partial f}{\partial q} \bigg|_{\mu_q} \right)^2 \sigma_q^2 \]  

(11)

It must be noted that the values \( \mu_z \) and \( \sigma_z^2 \) are only approximations to the true distribution of the values of \( z \).

### 2 n-D Case

In this section, we will concentrate on the situation described in the introduction, namely when \( q \) depends on multiple random variables.

The Taylor expansion around the values \( \mu^{(1)}, \mu^{(2)}, \ldots \) is given by,

\[ z = f(q^{(1)}, q^{(2)}, \ldots) \simeq f(\mu^{(1)}, \mu^{(2)}, \ldots) + \sum_{i=1}^{m} \frac{\partial f}{\partial q^{(i)}}(q^{(i)} - \mu^{(i)}) \]  

(12)

where for clarity we omit the explicit evaluation of the derivative at the mean value. We again compute the mean and variance of the \( z \) values using Equations 2 and 3,

\[ \mu_z = \frac{1}{n} \sum_{i=1}^{n} \left[ f(\mu^{(1)}, \mu^{(2)}, \ldots) + \sum_{j=1}^{m} \frac{\partial f}{\partial q^{(j)}}(q^{(j)} - \mu^{(j)}) \right] \]  

(13)

\[ = \frac{1}{n} \sum_{i=1}^{n} f(\mu^{(1)}, \mu^{(2)}, \ldots) + \frac{1}{n} \sum_{j=1}^{m} \frac{\partial f}{\partial q^{(j)}} \sum_{i=1}^{n} (q^{(j)} - \mu^{(j)}) \]  

(14)

\[ = f(\mu^{(1)}, \mu^{(2)}, \ldots) \]  

(15)

since by the definition of the mean, \( \sum_{i=1}^{n} (q^{(j)} - \mu^{(j)}) = 0 \) for each \( j \).

The variance of \( P(z) \) is,

\[ \sigma_z^2 = \frac{1}{n} \sum_{i=1}^{n} \left( f(\mu^{(1)}, \mu^{(2)}, \ldots) + \sum_{j=1}^{m} \frac{\partial f}{\partial q^{(j)}}(q^{(j)} - \mu^{(j)}) - \mu_z \right)^2 \]  

(16)

\[ = \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{m} \frac{\partial f}{\partial q^{(j)}}(q^{(j)} - \mu^{(j)}) \right)^2 \]  

(17)

\[ = \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{m} \frac{\partial f}{\partial q^{(j)}}(q^{(j)} - \mu^{(j)}) \sum_{k=1}^{m} \frac{\partial f}{\partial q^{(k)}}(q^{(k)} - \mu^{(k)}) \right), \]  

(18)
where we have expanded the square in Equation 17 by subscripting the sum with a \( j \) and \( k \). Expanding the multiplication of sums we have,

\[
\sigma_z^2 = \frac{1}{n} \sum_{i=1}^{n} \left[ \sum_{j=k=1}^{m} \left( \frac{\partial f}{\partial q^{(j)}} \right)^2 (q_i^{(j)} - \mu^{(j)})^2 + \right. \\
\left. \sum_{j=1}^{m} \sum_{k \neq j} \frac{\partial f}{\partial q^{(j)}} \frac{\partial f}{\partial q^{(k)}} (q_i^{(j)} - \mu^{(j)})(q_i^{(k)} - \mu^{(k)}) \right] 
\]

where the first term in the bracket comes from when \( j = k \) and the second term comes from the cross terms when \( j \neq k \). Now moving the sum over \( n \) into the brackets we have,

\[
\sigma_z^2 = \sum_{j=1}^{m} \left( \frac{\partial f}{\partial q^{(j)}} \right)^2 \frac{1}{n} \sum_{i=1}^{n} (q_i^{(j)} - \mu^{(j)})^2 + \\
2 \sum_{j=1}^{m} \sum_{k \neq j} \frac{\partial f}{\partial q^{(j)}} \frac{\partial f}{\partial q^{(k)}} \frac{1}{n} \sum_{i=1}^{n} (q_i^{(j)} - \mu^{(j)})(q_i^{(k)} - \mu^{(k)}) 
\]

\[
= \sum_{j=1}^{m} \left( \frac{\partial f}{\partial q^{(j)}} \right)^2 \sigma^{(j)2} + \sum_{j=1}^{m} \sum_{k \neq j} \frac{\partial f}{\partial q^{(j)}} \frac{\partial f}{\partial q^{(k)}} \frac{1}{n} \sum_{i=1}^{n} (q_i^{(j)} - \mu^{(j)})(q_i^{(k)} - \mu^{(k)}) 
\]

If we define \( \sigma_{jk}^2 \), the covariance, to be,

\[
\sigma_{jk} = \frac{1}{n} \sum_{i=1}^{n} (q_i^{(j)} - \mu^{(j)})(q_i^{(k)} - \mu^{(k)}) 
\]

then Equation 21 becomes,

\[
\sigma_z^2 = \sum_{j=1}^{m} \left( \frac{\partial f}{\partial q^{(j)}} \right)^2 \sigma^{(j)2} + \sum_{j=1}^{m} \sum_{k \neq j} \frac{\partial f}{\partial q^{(j)}} \frac{\partial f}{\partial q^{(k)}} \sigma_{jk}^2 
\]

### 2.1 2D Case

As a concrete example, let’s look at the case where \( z \) depends on two variables. In this case, \( m = 2 \) and the variance becomes,

\[
\sigma_z^2 = \left( \frac{\partial f}{\partial q^1} \right)^2 \sigma^{(1)2} + \left( \frac{\partial f}{\partial q^2} \right)^2 \sigma^{(2)2} + \frac{\partial f}{\partial q^{(1)}} \frac{\partial f}{\partial q^{(2)}} \sigma_{12}^2 + \frac{\partial f}{\partial q^{(2)}} \frac{\partial f}{\partial q^{(1)}} \sigma_{21}^2 
\]
Since $\sigma_{12} = \sigma_{21}$, we can rewrite this as,

$$\sigma_z^2 = \left( \frac{\partial f}{\partial q^1} \right)^2 \sigma^{(1)2} + \left( \frac{\partial f}{\partial q^2} \right)^2 \sigma^{(2)2} + 2 \frac{\partial f}{\partial q^{(1)}} \frac{\partial f}{\partial q^{(2)}} \sigma_{12}^2$$  \hspace{1cm} (25)

If we substitute $x$ and $y$ for $q^1$ and $q^2$ we have the more familiar form for the variance of $z$,

$$\sigma_z^2 = \left( \frac{\partial f}{\partial x} \right)^2 \sigma^{(x)2} + \left( \frac{\partial f}{\partial y} \right)^2 \sigma^{(y)2} + 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \sigma_{xy}^2$$  \hspace{1cm} (26)