$\Delta\chi^2$ Values

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1 Introduction

This is by no means an explanation of how to apply the $\Delta\chi^2$ technique, but rather an explanation of how numbers such as $\Delta\chi^2$ = 2.30 for 68% probability (2 degrees of freedom) arise.

2 One Dimensional Case

This is somewhat of a simple example, but it is a useful comparison for the two dimensional case discussed in the next section. Consider a random variable X that is distributed as,

$$p(X|\,\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-0.5\left(\frac{X-\mu}{\sigma}\right)^2}. \tag{1}$$

Now consider the intersection of this curve with a line of constant probability (in the more general case a hypersurface). The intersection is defined by the equation,

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-0.5\left(\frac{X-\mu}{\sigma}\right)^2} = P_0, \tag{2}$$

which is equivalent to,

$$\left(\frac{X-\mu}{\sigma}\right)^2 = -2\ln\left(\sqrt{2\pi}\sigma P_0\right) \tag{3}$$

$$= C, \tag{4}$$

if we solve Equation 2 for the exponent. As we might expect from our intuition, this last equation describes two points whose coordinates are given by $\mu \pm \sigma\sqrt{C}$.

Now the probability of observing a value X within this range R is simply the integral of the probability distribution over R or,

$$P\left(\mu - \sigma\sqrt{C} \leq X \leq \mu + \sigma\sqrt{C}\right) = \int_R p(X|\,\mu,\sigma) \tag{5}$$

But using Equation 4 this is equivalent to,

$$P\left(\left(\frac{X-\mu}{\sigma}\right)^2 \leq C\right) = \int_R p(X|\,\mu,\sigma) \tag{6}$$

$$P\left(Z \leq C\right) = \int_R p(X|\,\mu,\sigma) \tag{7}$$

$$P\left(\chi_1^2 \leq C\right) = \int_R p(X|\,\mu,\sigma), \tag{8}$$

where Z is the standard normal deviate (which is distributed as $\mathcal{N}(0,1)$) and χ_1^2 is the χ^2 distribution with 1 degree of freedom.

We can now set the RHS of Equations 7 or 8 to whatever integrated probability we desire, and solve them for C. Substituting C into Equation 4 defines the region which contains the desired integrated probability. For example, C=1.0 (68.3%), 2.71 (90%), 3.84 (95%), 4.00 (95.4%), 6.63 (99%), 9.00 (99.73%), and 15.1 (99.99%).

3 Two Dimensional Case

Consider two random variables X_1 and X_2 that are both independent and normally distributed as,

$$X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2) \tag{9}$$

$$X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2) \tag{10}$$

The joint probability distribution of X_1 and X_2 is,

$$p(X_1, X_2|\,\mu_1, \sigma_1^2, \mu_2, \sigma_2^2) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-0.5}\left\{\left(\frac{X_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{X_2-\mu_2}{\sigma_2}\right)^2\right\} \tag{11}$$

We can now proceed as we did in §2 by identifying the region defined by the intersection of, in the 2D case, a surface of constant probability with this distribution. Obviously,

$$\left(\frac{X_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{X_2-\mu_2}{\sigma_2}\right)^2 = -2\ln\left(2\pi\sigma_1\sigma_2 P_0\right) \tag{12}$$

$$\left(\frac{X_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{X_2-\mu_2}{\sigma_2}\right)^2 = C \tag{13}$$

This is the equation of an ellipse centered μ_1 and μ_2.

The probability that X_1 and X_2 fall within the region (ellipse) R defined by Equation 13 is,

$$P\left(\left(\frac{X_1-\mu_1}{\sigma_1}\right)^2+\left(\frac{X_2-\mu_2}{\sigma_2}\right)^2 \leq C\right) = \int_R p(X_1, X_2|\,\mu_1, \sigma_1^2, \mu_2, \sigma_2^2) \tag{14}$$

$$P(Z_1+Z_2 \leq C) = \int_R p(X_1, X_2|\,\mu_1, \sigma_1^2, \mu_2, \sigma_2^2) \tag{15}$$

$$P(\chi_2^2 \leq C) = \int_R p(X_1, X_2|\,\mu_1, \sigma_1^2, \mu_2, \sigma_2^2) \tag{16}$$

Again we can now set the RHS of Equation 16 to whatever integrated probability we desire, and solve it for C. Substituting C into Equation 13 then defines the region which contains the desired integrated probability. For example, C=2.30 (68.3%), 4.61 (90%), 5.99 (95%), 6.17 (95.4%), 9.21 (99%), 11.8 (99.73%), and 18.4 (99.99%).