1 Introduction

This is by no means an explanation of how to apply the $\Delta \chi^2$ technique, but rather an explanation of how numbers such as $\Delta \chi^2 = 2.30$ for 68% probability (2 degrees of freedom) arise.

2 One Dimensional Case

This is somewhat of a simple example, but it is a useful comparison for the two dimensional case discussed in the next section. Consider a random variable $X$ that is distributed as,

$$p(X|\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-0.5\left(\frac{X-\mu}{\sigma}\right)^2}.$$  (1)

Now consider the intersection of this curve with a line of constant probability (in the more general case a hypersurface). The intersection is defined by the equation,

$$\frac{1}{\sqrt{2\pi\sigma}} e^{-0.5\left(\frac{X-\mu}{\sigma}\right)^2} = P_0,$$  (2)

which is equivalent to,

$$\left(\frac{X-\mu}{\sigma}\right)^2 = -2\ln\left(\sqrt{2\pi\sigma P_0}\right)$$  (3)

$$= C,$$  (4)

if we solve Equation 2 for the exponent. As we might expect from our intuition, this last equation describes two points whose coordinates are given by $\mu \pm \sigma \sqrt{C}$.

Now the probability of observing a value $X$ within this range $R$ is simply the integral of the probability distribution over $R$ or,

$$P\left(\mu - \sigma \sqrt{C} \leq X \leq \mu + \sigma \sqrt{C}\right) = \int_R p(X|\mu,\sigma)$$  (5)

But using Equation 4 this is equivalent to,
\[ P \left( \left( \frac{X - \mu}{\sigma} \right)^2 \leq C \right) = \int_R p(X|\mu,\sigma) \]  \hspace{2cm} (6)

\[ P (Z \leq C) = \int_R p(X|\mu,\sigma) \]  \hspace{2cm} (7)

\[ P \left( \chi^2 \leq C \right) = \int_R p(X|\mu,\sigma), \]  \hspace{2cm} (8)

where \( Z \) is the standard normal deviate (which is distributed as \( \mathcal{N}(0, 1) \)) and \( \chi^2 \) is the \( \chi^2 \) distribution with 1 degree of freedom.

We can now set the RHS of Equations 7 or 8 to whatever integrated probability we desire, and solve them for \( C \). Substituting \( C \) into Equation 4 defines the region which contains the desired integrated probability. For example, \( C=1.0 \) (68.3%), 2.71 (90%), 3.84 (95%), 4.00 (95.4%), 6.63 (99%), 9.00 (99.73%), and 15.1 (99.99%).

3 Two Dimensional Case

Consider two random variables \( X_1 \) and \( X_2 \) that are both independent and normally distributed as,

\[ X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2) \]  \hspace{2cm} (9)

\[ X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2) \]  \hspace{2cm} (10)

The joint probability distribution of \( X_1 \) and \( X_2 \) is,

\[ p(X_1, X_2|\mu_1, \sigma_1^2, \mu_2, \sigma_2^2) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-0.5 \left\{ \left( \frac{X_1 - \mu_1}{\sigma_1} \right)^2 + \left( \frac{X_2 - \mu_2}{\sigma_2} \right)^2 \right\}} \]  \hspace{2cm} (11)

We can now proceed as we did in §2 by identifying the region defined by the intersection of, in the 2D case, a surface of constant probability with this distribution. Obviously,

\[ \left( \frac{X_1 - \mu_1}{\sigma_1} \right)^2 + \left( \frac{X_2 - \mu_2}{\sigma_2} \right)^2 = -2 \ln (2\pi\sigma_1\sigma_2 P_0) \]  \hspace{2cm} (12)

\[ \left( \frac{X_1 - \mu_1}{\sigma_1} \right)^2 + \left( \frac{X_2 - \mu_2}{\sigma_2} \right)^2 = C \]  \hspace{2cm} (13)

This is the equation of an ellipse centered \( \mu_1 \) and \( \mu_2 \).
The probability that $X_1$ and $X_2$ fall within the region (ellipse) $R$ defined by Equation 13 is,

$$P \left( \left( \frac{X_1 - \mu_1}{\sigma_1} \right)^2 + \left( \frac{X_2 - \mu_2}{\sigma_2} \right)^2 \leq C \right) = \int_{R} p(X_1, X_2 | \mu_1, \sigma_1^2, \mu_2, \sigma_2^2) \quad (14)$$

$$P(Z_1 + Z_2 \leq C) = \int_{R} p(X_1, X_2 | \mu_1, \sigma_1^2, \mu_2, \sigma_2^2) \quad (15)$$

$$P(\chi_2^2 \leq C) = \int_{R} p(X_1, X_2 | \mu_1, \sigma_1^2, \mu_2, \sigma_2^2) \quad (16)$$

Again we can now set the RHS of Equation 16 to whatever integrated probability we desire, and solve it for $C$. Substituting $C$ into Equation 13 then defines the region which contains the desired integrated probability. For example, $C=2.30$ (68.3%), 4.61 (90%), 5.99 (95%), 6.17 (95.4%), 9.21 (99%), 11.8 (99.73%), and 18.4 (99.99%).